EXOTIC CHARACTERISTIC CLASSES OF QUATERNIONIC BUNDLES

BY

IZU VAISMAN

[a]*Department of Mathematics, University of Haifa, Haifa, Israel*

ABSTRACT

For a C^∞ quaternionic vector bundle, the odd-dimensional real Chern classes vanish, and this allows for a construction of secondary (exotic) characteristic classes associated with a pair of quaternionic structures of a given complex vector bundle. This construction is then applied to obtain exotic characteristic classes associated with an automorphism β of the holomorphic tangent bundle of a Kähler manifold. These results are the complex analoga of those given for the higher order Maslov classes in [V2].

The present paper is the result of a quest for new applications of the theory of secondary (exotic) characteristic classes. This led us to notice that our construction of higher order Maslov classes [V2] has a complex version which yields exotic characteristic classes associated with a pair of quaternionic structures of a given C^∞ complex vector bundle, and implicitly, with a pair of complex symplectic structures defined on a complex vector bundle. After a description of quaternionic structures on complex vector bundles in Section 1, we construct the exotic classes mentioned above in Section 2. Finally, in Section 3, we give an application to the case where the complex vector bundle is the holomorphic tangent bundle of a manifold V which itself is the holomorphic cotangent bundle of a Kähler manifold M. This bundle has a canonical symplectic structure. Then, if M is also endowed with an automorphism β of its holomorphic tangent bundle, a second complex symplectic structure is obtained. This leads to exotic characteristic classes which retract to

Received March 1, 1989

M, and are invariants of (M, β). Differential forms which represent these invariants are constructed.

1. Quaternionic structures on complex vector bundles

We shall denote by $\mathbf{Q}$ the field of the quaternions, by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ its usual basis over $\mathbb{R}$, and we shall also look at $(\mathbf{1}, \mathbf{j})$ as a basis of $\mathbf{Q}$ over $\mathbb{C}$, where $\mathbb{C}$ has the basis $(\mathbf{1}, \mathbf{i})$ over $\mathbb{R}$. We send the reader to [D], for instance, for an exposition on quaternions, and we use here the results exposed in [D].

Particularly, a quaternionic vector space may be defined directly over $\mathbf{Q}$, and we see it with the right-hand side scalar multiplication, or we may define it as a complex vector space E of an even dimension $2n$ over $\mathbb{C}$, which is endowed with an *antilinear* mapping $J: E \to E$ (i.e., $J(\mathbf{v}\lambda) = J(\mathbf{v})\bar{\lambda}$, $\mathbf{v} \in E$, $\lambda \in \mathbb{C}$), such that $J^2 = -\mathrm{Id}$. Then J plays the role of right multiplication by $\mathbf{j}$.

It is easy to see that the set $\mathcal{Q}(E)$ of all the quaternionic structures of E is given by

$$(1.1) \qquad \mathcal{Q}(E) \approx \mathrm{Gl}(2n, \mathbb{C})/\mathrm{Gl}(n, \mathbf{Q}).$$

Another noticeable fact is that E has a class of *adapted bases* defined by having the form

$$(1.2) \qquad \mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{i^*} = J\mathbf{e}_{1^*}, \ldots, \mathbf{e}_{n^*} = J\mathbf{e}_n,$$

where $i^* = i + n$. These are generated by the bases $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of (E, J) over $\mathbf{Q}$.

Furthermore, a *quaternionic Hermitian metric* of E is an $\mathbb{R}$-bilinear mapping $b: E \times E \to \mathbf{Q}$ such that

$$(1.3) \qquad \begin{aligned} &b(\mathbf{x}\lambda, \mathbf{y}) = \bar{\lambda} b(\mathbf{x}, \mathbf{y}), \qquad b(\mathbf{x}, \mathbf{y}\lambda) = b(\mathbf{x}, \mathbf{y})\lambda, \\ &b(\mathbf{x}, \mathbf{y}) = \overline{b(\mathbf{y}, \mathbf{x})}, \qquad b(\mathbf{x}, \mathbf{x}) > 0 \qquad \text{if } \mathbf{x} \neq 0, \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in E$, $\lambda \in \mathbf{Q}$ and the bar means conjugation in $\mathbf{Q}$.

If we put

$$(1.4) \qquad b(\mathbf{y}, \mathbf{x}) = h(\mathbf{x}, \mathbf{y}) + \mathbf{j}\omega(\mathbf{x}, \mathbf{y}),$$

it follows easily that h is a Hermitian metric on E seen as a complex vector bundle, and ω is a complex bilinear skew symmetric 2-form on E, such that

$$(1.5) \qquad h(J\mathbf{x}, J\mathbf{y}) = \overline{h(\mathbf{x}, \mathbf{y})}, \qquad \omega(J\mathbf{x}, J\mathbf{y}) = \overline{\omega(\mathbf{x}, \mathbf{y})},$$

$$\omega(\mathbf{x}, \mathbf{y}) = -\overline{h(J\mathbf{x}, \mathbf{y})}, \qquad h(\mathbf{x}, \mathbf{y}) = \overline{\omega(J\mathbf{x}, \mathbf{y})}. \tag{1.6}$$

Moreover, using (1.6) we see that ω is nondegenerate, i.e., ω is a symplectic form on E.

Conversely, if (E, J) is given either a complex Hermitian metric h which satisfies the first relation (1.5) or a complex symplectic form ω which satisfies the second relation (1.5), then we may use (1.6), (1.4) to define a quaternionic Hermitian metric on E. In both cases we shall say that J is *compatible* with h and ω, respectively. In case we start with ω, the condition that the corresponding h be positive definite must be added separately, and then we also say that J is *positive* for ω. If a J-compatible h is given, we shall also call h a *quaternionic Hermitian metric* on (E, J) in spite of the fact that it is rather b of (1.4) which is the metric. Then ω will be the *fundamental symplectic form of* h (not to be confused with a closed differential symplectic form on a manifold).

If (E^{2n}, J, h) is a quaternionic Hermitian vector space, the Gramm–Schmidt procedure yields **Q**-bases $(\mathbf{e}_i)_{i=1}^n$ where b has the canonical form

$$b(\mathbf{e}_i\lambda^i, \mathbf{e}_j\eta^j) = \sum_{i=1}^{n} \bar{\lambda}^i\eta^i \tag{1.7}$$

(the Einstein sum convention is always used in this paper, if possible). Such bases, as well as the corresponding adapted bases (1.2) are called *quaternionic unitary bases* and, with respect to such bases, both h and ω assume a corresponding canonical form. This justifies the well known relation $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap U(2n)$, where $\mathrm{Sp}(2n, \mathbb{C})$ is the complex symplectic group, $U(2n)$ is the unitary group and $\mathrm{Sp}(n)$ is the *quaternionic unitary group* (for which many authors also use the name symplectic group).

It is easy to understand that any complex Hermitian vector space (E^{2n}, h) has compatible quaternionic structures J, and the set of these structures is

$$\mathscr{J}(E^{2n}, h) = U(2n)/\mathrm{Sp}(n). \tag{1.8}$$

Similarly, a complex symplectic vector space (E^{2n}, ω) has positive compatible quaternionic structures J and their set is

$$\mathscr{J}(E^{2n}, \omega) = \mathrm{Sp}(2n, \mathbb{C})/\mathrm{Sp}(n). \tag{1.9}$$

In both cases we obtain such a J by taking a canonical basis (unitary and symplectic, respectively), and putting it under the form $(\mathbf{e}_i, J\mathbf{e}_i)$.

Moreover we have

PROPOSITION 1.1. *Let (E^{2n}, ω) be a complex symplectic vector space and γ a Hermitian metric on E. Then, a canonical procedure to associate a positive compatible quaternionic structure J to γ exists. Every such structure J on (E^{2n}, ω) is associated in this way with certain metrics γ, and two structures J_1, J_2 are connected by a differentiable path J_t $(0 \leqq t \leqq 1)$ in $\mathcal{J}(E^{2n}, \omega)$.*

The proof is like that for real symplectic spaces (e.g., [V2]).

Namely, if γ is given, we take the unique antilinear isomorphism $a : E \to E$ defined by

$$\gamma(a\mathbf{x}, \mathbf{y}) = -\overline{\omega(\mathbf{x}, \mathbf{y})}.$$

a satisfies

$$\gamma(a^2\mathbf{x}, \mathbf{y}) = -\overline{\gamma(a\mathbf{x}, a\mathbf{y})} = \gamma(\mathbf{x}, a^2\mathbf{y}),$$

hence a^2 is negative and self-adjoint and $-a^2$ has a unique positive square root ρ which commutes with a, where ρ is a linear isomorphism of E.

Then $J = a\rho^{-1} = \rho^{-1}a$ is the quaternionic structure requested.

Conversely, every J is obtained by the above procedure at least from the corresponding h of (1.6). Then, if $J_{1,2}$ are associated with $\gamma_{1,2}$ we shall associate J_t with $(1-t)\gamma_1 + t\gamma_2$. Q.e.d.

Another important fact is

PROPOSITION 1.2. *Let (E^{2n}, ω) be a complex symplectic vector space, and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a maximal system of independent pairwise ω-orthogonal vectors of E. Then, for any positive ω-compatible quaternionic structure J of (E, ω) the vectors $(\mathbf{e}_i, J\mathbf{e}_i)$ form an adapted complex basis.*

Indeed, under the hypotheses we have $\operatorname{rank}_{\mathbf{C}}\omega(\mathbf{e}, J\mathbf{e}) = 2n$. Q.e.d.

The structures described above extend straightforwardly to vector bundles, where J, b, h, ω etc. become corresponding vector bundle cross sections. Notice that a complex vector bundle does not necessarily have a quaternionic structure J, even if the fibers are even dimensional.

In this paper our framework is the C^∞-category, hence all our bundles and cross sections are assumed to be of class C^∞. Particularly, one might look at the holomorphic tangent bundles of almost complex manifolds, and then a quaternionic structure of the bundle is an *almost-quaternionic structure* of the manifold. Such structures were studied by various authors, particularly by E. Bonan [B].

In order to discuss characteristic classes we must look at quaternionic connections of a quaternionic vector bundle (E, J) over a manifold M^m. (See [B] for a study of important connections of an almost quaternionic manifold.) If $(\mathbf{e}_i)$ is a local field of bases over $\mathbf{Q}$, a quaternionic connection ∇ has local equations,

$$\nabla \mathbf{e}_i = \mathbf{e}_h \theta_i^h, \tag{1.10}$$

where θ_i^h are $\mathbf{Q}$-valued 1-forms, and may be written as

$$\theta_i^h = \alpha_i^h + \mathbf{j}\beta_i^h, \tag{1.11}$$

where α_i^h, β_i^h are $\mathbb{C}$-valued 1-forms. It follows that a complex connection on E is quaternionic iff its equations with respect to adapted bases are of the form

$$\begin{aligned} \nabla \mathbf{e}_i &= \alpha_i^h \mathbf{e}_h + \beta_i^h (J\mathbf{e}_h), \\ \nabla (J\mathbf{e}_i) &= -\bar{\beta}_i^h \mathbf{e}_h + \bar{\alpha}_i^h (J\mathbf{e}_h). \end{aligned} \tag{1.12}$$

If the bundle also has a quaternionic Hermitian metric b, and if $(\mathbf{e}_i)$ is a $\mathbf{Q}$-unitary basis, the connection ∇ preserves this metric iff

$$\theta_i^h + \bar{\theta}_h^i = 0 \tag{1.13}$$

or, equivalently,

$$\alpha_i^h + \bar{\alpha}_h^i = 0, \qquad \beta_i^h - \beta_h^i = 0. \tag{1.14}$$

The curvature Ξ of ∇ has local forms

$$\begin{pmatrix} A_i^h & B_i^h \\ -\bar{B}_i^h & \bar{A}_i^h \end{pmatrix}, \tag{1.15}$$

and in the case of a metric connection and $\mathbf{Q}$-unitary bases we also have

$$A_i^h + \bar{A}_h^i = 0, \qquad B_i^h = B_h^i. \tag{1.16}$$

2. Characteristic classes

Let E be a complex vector bundle. Then its basic characteristic classes are the Chern classes $c_k(E)$. Here we consider only real cohomology classes and, therefore, we shall represent $c_k(E)$ by the *Chern form* (e.g., [KN])

$$\mathfrak{C}_k(\Omega) = \frac{(\sqrt{-1})^k}{(2\pi)^k k!} \, \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} \Omega_{j_1}^{i_1} \wedge \cdots \wedge \Omega_{j_k}^{i_k}, \tag{2.1}$$

where Ω is the curvature of a Hermitian connection of E, and $\delta^{...}_{...}$ is the multiindex Kronecker symbol.

Now if E admits a quaternionic structure, we have the following important known result (e.g., [KN], [GHV])

PROPOSITION 2.1. *If E has a quaternionic structure J then $c_{2s+1}(E)=0$* $(s=0,1,2,\ldots)$.

PROOF. Let b be a corresponding quaternionic Hermitian metric with the Hermitian part h, and the symplectic part ω as in (1.4), and let $(\mathbf{e}_i, J\mathbf{e}_i)$ be a unitary adapted basis of E. Then two duality relations are defined, by h and ω respectively, between E and $E^*=$ the dual bundle of E. By h-duality, $(\mathbf{e}_i, J\mathbf{e}_i)$ goes to $(\varepsilon^i, -\varepsilon^i\circ J)$, where $\varepsilon^i(\mathbf{e}_k)=\delta^i_k$, and then by ω-duality $(\varepsilon^i, -\varepsilon^i\circ J)$ goes to $(\mathbf{f}_i=-J\mathbf{e}_i, \mathbf{f}_{i^*}=\mathbf{e}_i)$ (see (1.2) for notation).

Now let ∇ of (1.10), (1.11) be a quaternionic metric connection of E. Then with respect to $(\mathbf{f}_i, \mathbf{f}_{i^*})$ ∇ has the local equations

$$\text{(2.2)}\qquad \begin{aligned} \nabla\mathbf{f}_i &= \bar{\alpha}_i^h\mathbf{f}_h + \bar{\beta}_i^h\mathbf{f}_{h^*},\\ \nabla\mathbf{f}_{i^*} &= -\beta_i^h\mathbf{f}_h + \alpha_i^h\mathbf{f}_{h^*}, \end{aligned}$$

and it follows that the curvature matrix of ∇ with respect to these bases is $\bar{\Xi}$, where Ξ is the curvature matrix (1.15). Remember also that this matrix satisfies (1.16), which amounts to $\bar{\Xi}=-{}^t\Xi$ (t = transposition).

Furthermore, since the forms $\mathfrak{C}_k$ of (2.1) do not depend on the choice of the local bases, we deduce from the above mentioned facts and from (2.1) that

$$\text{(2.3)}\qquad \mathfrak{C}_k(\Xi)=\mathfrak{C}_k(\bar{\Xi})=(-1)^k\mathfrak{C}_k({}^t\Xi)=(-1)^k\mathfrak{C}_k(\Xi). \qquad \text{Q.e.d.}$$

REMARK. *From Proposition* 1.1 *it follows that Proposition* 2.1 *holds for any complex symplectic vector bundle* (E,ω).

Not only the content but the proof of Proposition 2.1 is in fact important since it gives us a class of connections which are such that $\mathfrak{C}_{2s+1}$ vanishes for the curvature of these connections. This allows for the application of the general theory of secondary characteristic classes in its Chern–Simmons–Bott–Lehmann variant (e.g., [V2]) (see [KT], pp. 151–152, for the Kamber–Tondeur approach). Since only a certain sequence of these classes generate all of them, we shall content ourselves with the description of the classes of this sequence.

Namely, let $\pi: E\to M$ be a complex vector bundle which is endowed with two quaternionic structures J_1, J_2. Let ∇^1, ∇^2 be quaternionic metric connections on (E, J_1), (E, J_2) respectively, and let be

$$\nabla^a \mathbf{c}_i = \gamma_i^{aj} \mathbf{c}_j \qquad (a = 1, 2; \quad i, j = 1, \ldots, 2n), \tag{2.4}$$

the local equations of these connections with respect to arbitrary local complex bases $(\mathbf{c}_i)$ of E. Define

$$\begin{aligned} &\mathfrak{T}_k(\nabla^1, \nabla^2) \\ &= \frac{(\sqrt{-1})^k}{(2\pi)^k (k-1)!} \int_{t=0}^{t=1} (\delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} \sigma_{j_1}^{i_1} \wedge \Pi_{tj_2}^{i_2} \wedge \cdots \wedge \Pi_{tj_k}^{i_k}) dt, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\sigma_j^i = \gamma_j^{2i} - \gamma_j^{1i}, \\ &\Pi_{tj}^i = (1-t)\Pi_{0j}^i + t\Pi_{1i}^j + t(1-t)\sigma_j^h \wedge \sigma_h^i, \end{aligned} \tag{2.6}$$

and Π_0, Π_1 are the curvature forms of ∇^1, ∇^2, respectively.

Then a classical fundamental formula (e.g. [KN]) tells us that

$$d\mathfrak{T}_k(\nabla^1, \nabla^2) = \mathfrak{C}_k(\Pi_2) - \mathfrak{C}_k(\Pi_1), \tag{2.7}$$

and, therefore, since the ∇^a are metric connections and in view of the proof of Proposition 2.1, we get

$$d\mathfrak{T}_{2s+1}(\nabla^1, \nabla^2) = 0 \qquad (s = 0, 1, 2, \ldots). \tag{2.8}$$

Accordingly, we obtain cohomology classes

$$\xi_{2s+1}(J^1, J^2) = [\mathfrak{T}_{2s+1}(\nabla^1, \nabla^2)] \in H^{4s+1}(M, \mathbb{R}), \tag{2.9}$$

and we call them the *exotic classes* of (J^1, J^2). The notation is correct since a usual homotopy argument (e.g., [V2], p. 167) shows that the exotic classes do not depend on the choice of the connections ∇^1, ∇^2. (The convex combination of two different choices is quaternionic and metric for every $0 \leqq t \leqq 1$.)

A similar homotopy argument based on the last assertion of Proposition 1.1 (see the proof of Theorem 4.4.25 of [V2]) shows that if the complex vector bundle E has two complex symplectic structures ω_a $(a = 1, 2)$ exotic classes

$$\xi_{2s+1}(\omega_1, \omega_2) \in H^{4s+1}(M, \mathbb{R}) \tag{2.10}$$

may be defined as being the classes (2.9) of quaternionic structures J^a which are positive and compatible with respect to ω_a $(a = 1, 2)$. These classes do not depend on the choice of the compatible structures J^a. In the same way, it is clear that if ω_1, ω_2 can be related by a differentiable 1-parameter family $\omega(t)$ of complex symplectic structures ($0 \leqq t \leqq 1$, $\omega(0) = \omega_1$, $\omega(1) = \omega_2$) then $\xi_{2s+1}(\omega_1, \omega_2) = 0$. Hence we proved

PROPOSITION 2.2. *Any pair (ω_1, ω_2) of complex symplectic structures on a vector bundle E has a sequence of associated exotic characteristic classes* (2.10) *which vanish if ω_1 and ω_2 are differentiably homotopically related via complex symplectic structures ω_t.*

Finally, let us mention another interesting property of the exotic classes (2.9), (2.10), namely

PROPOSITION 2.3. *If ω_a $(a = 1, 2, 3)$ are three complex symplectic structures on E then one has*

(a) $\xi_{2s+1}(\omega_1, \omega_2) = -\xi_{2s+1}(\omega_2, \omega_1)$,

(b) $\xi_{2s+1}(\omega_1, \omega_2) + \xi_{2s+1}(\omega_2 \omega_3) + \xi_{2s+1}(\omega_3, \omega_1) = 0$.

The proof is the same as for Proposition 4.4.27 of [V2].

3. Cotangent bundles of Kähler manifolds

The simplest example of a quaternionic bundle is given by the holomorphic tangent bundle $T_{\text{hol}}V$ (i.e., the bundle of the tangent vectors of the complex type $(1, 0)$ of V) of a manifold $V = T^*_{\text{hol}}M$, where M^n is any complex manifold. In this case, computations analogous to those of Section 4.5 of [V2] can be made.

Namely, V has a well known *canonical* complex symplectic structure $\omega_0 = -d\lambda$, where λ is the *complex Liouville form* given by

$$\lambda_\zeta(v) = \zeta(\pi_* \mathbf{v}), \qquad \zeta \in V, \quad \mathbf{v} \in (T_{\text{hol}}V)_\zeta, \tag{3.1}$$

and $\pi : V \to M$ is the natural projection. The integrability of the complex structure of M ensures that $d\lambda$ is of the complex type $(2, 0)$. Hereafter, and since we never use in this paper the real form of the tangent bundle TV, we shall denote $T_{\text{hol}}V$ by TV.

Accordingly, we shall have ω_0-compatible positive quaternionic structures on TV. If M has Kähler metrics, such structures can be easily obtained as follows. Let g be a Kähler metric on M, and D be its Levi-Civita connection. The Kählerian character of g ensures that D is compatible with the complex structure of M. Then we may write

$$TV = \mathscr{H} \oplus \mathscr{V}, \tag{3.2}$$

where $\mathscr{V}$ is the *holomorphic vertical subbundle* tangent to the fibers of V, and $\mathscr{H}$ is the *holomorphic horizontal subbundle* defined by the connection D. An antilinear endomorphism $J: TV \to TV$ with $J^2 = -\mathrm{Id}$ is then determined by asking $J/\mathscr{H}$ to be given by the composition

$$\mathscr{H} \underset{g}{\approx} \bar{\mathscr{H}}^* \underset{\omega_0}{\approx} \bar{\mathscr{V}} \underset{\text{conj}}{\approx} \mathscr{V}, \tag{3.3}$$

where g is lifted in a natural manner to $\mathscr{H}$, and it follows that J is ω_0-compatible and positive.

All these also follow from the following local computation. If z^h $(h = 1, \dots, n)$ are local complex coordinates on M, and ζ_h are corresponding natural covector coordinates, we see that

$$\lambda = \zeta_h dz^h, \qquad \omega_0 = dz^h \wedge d\zeta_h, \tag{3.4}$$

and that the subbundle $\mathscr{H}$ is defined by the equations

$$\theta_h = d\zeta_h - \Gamma^l_{hk}\zeta_l dz^k = 0, \tag{3.5}$$

where Γ^l_{hk} are the Christoffel symbols of g.

Hence, $\mathscr{H}$ has the local basis

$$Z_h = \frac{\partial}{\partial z^h} + \Gamma^l_{hk}\zeta_l \frac{\partial}{\partial \zeta_k}, \tag{3.6}$$

as well as the basis

$$Y^{\bar{h}} = g^{\bar{h}k} Z_k, \tag{3.7}$$

and the transition functions of these bases by the coordinate change $z'^h = z'^h(z^k)$ are

$$Z'_h = \frac{\partial z}{\partial z'^h} Z_k, \qquad Y'^{\bar{h}} = \frac{\partial \bar{z}'^h}{\partial \bar{z}^k} Y^{\bar{k}}. \tag{3.8}$$

Accordingly, (3.3) shows us that J is given by the local equations

$$J(Y^{\bar{h}}) = -\frac{\partial}{\partial \zeta_h}, \qquad J\left(\frac{\partial}{\partial \zeta_h}\right) = Y^{\bar{h}}, \tag{3.9}$$

and it is easy to check the ω_0-compatibility and positivity of J. The Hermitian metric

$$h(\mathbf{x}, \mathbf{y}) = \overline{\omega_0(J\mathbf{x}, \mathbf{y})} \qquad (\mathbf{x}, \mathbf{y} \in TV) \tag{3.10}$$

is determined by $h/\mathscr{H} = \pi^*g$, where g is seen as a Hermitian metric on $T_{\mathrm{hol}}M$ and by J-compatibility.

Finally, following again the real case [V2], we can write down a quaternionic metric connection ∇ on (TV, J, h). Namely, we shall ask ∇ to satisfy the conditions: (i) ∇-parallelism preserves $\mathscr{H}$ and $\mathscr{V}$, (ii) ∇-parallelism preserves J, (iii) $\nabla/\mathscr{H} = \pi^{-1}(D)$, and we get for ∇ the local equations

$$\nabla Y^{\bar{h}} = -(\pi^*\bar{\omega}_{\bar{k}}^{\bar{h}})Y^{\bar{k}}, \qquad \nabla \frac{\partial}{\partial \zeta_h} = -(\pi^*\omega_k^h)\frac{\partial}{\partial \zeta_k}. \tag{3.11}$$

It is easy to write down other complex symplectic forms on V also, for instance, the forms $\omega_0 + \pi^*F$, where F is a form of the complex type (2, 0) on M. But these latter forms are clearly homotopically deformable to ω_0, and the corresponding exotic classes are zero.

In order to get a more interesting situation we shall assume that $T_{\mathrm{hol}}M$ is endowed with a certain complex linear bundle automorphism β. Then TV has also the complex symplectic form

$$\omega(\beta) = -d_f(\beta^*\lambda), \tag{3.12}$$

where λ is the Liouville form (3.1), β is interpreted as a mapping $\beta: V \to V$, and d_f denotes the exterior differential along the leaves of the vertical foliation $\mathscr{V}$ of V, i.e., the projection of d on $\mathscr{V}^*$ defined by the decomposition $TV = \mathscr{H} \oplus \mathscr{V}$ (e.g., see [V1] for a precise definition of d_f). Since for $\beta = \mathrm{Id}$ we have $\omega(\beta) = \omega_0$, the exotic classes of ω_0 and $\omega(\beta)$ will distinguish the automorphisms β which cannot be homotopically deformed to Id. Moreover, because of Proposition 2.3, if β_1, β_2 yield different exotic classes, they are nonhomotopical to each other via automorphisms of $T_{\mathrm{hol}}M$. We proceed now with the computation of these classes.

It follows from (3.12), (3.1) and (3.5) that we have

$$\omega(\beta) = \beta_i^j dz^i \wedge \theta_j = \kappa^j \wedge \theta_j \qquad (\kappa^j \overset{\mathrm{def}}{=} \beta_i^j dz^i), \tag{3.13}$$

which shows that (κ^j, θ_j) is a canonical cobasis of $\omega(\beta)$, and

$$\tilde{Z}_j = \eta_j^k Z_k, \frac{\partial}{\partial \zeta_j} \qquad (\eta_j^k \beta_k^h = \delta_j^h) \tag{3.14}$$

is the corresponding dual basis (Z_k was defined in (3.6)). Furthermore, if we define

(3.15) $$\tilde{Y}^{\bar{k}} = g^{\bar{k}j}\tilde{Z}_j,$$

it follows that the equations

(3.16) $$\tilde{J}\tilde{Y}^{\bar{k}} = -\frac{\partial}{\partial\zeta_k}, \qquad \tilde{J}\frac{\partial}{\partial\zeta_k} = \tilde{Y}^k$$

provide us with an $\omega(\beta)$-compatible positive quaternionic structure $\tilde{J}$ on TV.

As a matter of fact, $\tilde{J}$ could be defined invariantly just as J has been defined in (3.3) but using now $\omega(\beta)$ instead of ω_0. A corresponding Hermitian metric $\tilde{h}$ will be associated with $(\omega(\beta), \tilde{J})$ by (1.6) and a quaternionic metric connection $\tilde{\nabla}$ with a definition similar to that of ∇ of (3.11) will exist and it will have the local equations

(3.17) $$\tilde{\nabla}\tilde{Y}^{\bar{h}} = -(\pi^*\bar{\omega}_{\bar{k}}^{\bar{h}})\tilde{Y}^{\bar{k}}, \qquad \tilde{\nabla}\frac{\partial}{\partial\zeta_h} = -(\pi^*\omega_k^h)\frac{\partial}{\partial\zeta_k}.$$

Now we have to compute differential forms as defined in formulas (2.6), (2.5). First of all, let us express $\tilde{\nabla}$ and ∇ with respect to the same bases, say $(Y^{\bar{h}}, \partial/\partial\zeta_k)$ of (3.11). ∇ is given by (3.11). Then (3.7) and (3.15) yield

(3.18) $$\tilde{Y}^{\bar{k}} = \eta^{\bar{k}}_{\cdot\bar{s}}Y^{\bar{s}}, \qquad Y^{\bar{k}} = \beta^{\bar{k}}_{\cdot\bar{s}}\tilde{Y}^{\bar{s}},$$

where

(3.19) $$\beta^{\bar{k}}_{\cdot\bar{s}} = g^{\bar{k}h}\beta_{h\cdot}^{\;l}g_{l\bar{s}}, \qquad \beta_{h\cdot}^{\;l} = \beta_h^l,$$

and similar formulas are used for η. This is the usual index tensorial technique, and if we go on with it we get

(3.20) $$\tilde{\nabla}Y^{\bar{h}} = (\eta^{\bar{l}}_{\cdot\bar{k}}D\beta^{\bar{h}}_{\cdot\bar{l}} - \bar{\omega}_{\bar{k}}^{\bar{h}})Y^{\bar{k}}, \qquad \tilde{\nabla}\frac{\partial}{\partial\zeta_h} = -(\omega_k^h)\frac{\partial}{\partial\zeta_k},$$

where we agreed to identify forms on M with their π^* — lift to V.

Now, from (2.5), (2.6), (3.11) and (3.20) it follows that

(3.21) $$\xi_1(\omega_0, \omega(\beta)) = \frac{\sqrt{-1}}{2\pi}[\eta^{\bar{l}}_{\cdot\bar{k}}D\beta^{\bar{k}}_{\cdot\bar{l}}] \in H^1(V, \mathbb{R})$$

is the first one of the exotic classes which we wanted to compute.

At this point, let us notice that, since M is a retract of V, the cohomology classes $\xi_{2s+1}(\omega_0, \omega(\beta))$ on V retract to cohomology classes $\xi^*_{2s+1}(\omega_0, \omega(\beta)) \in H^{4s+1}(M, \mathbb{R})$, and we shall define the latter as the *exotic classes* in this case.

Particularly, $\xi_1^*(\omega_0, \omega(\beta))$ is again defined by the differential form in (3.21), but on M instead of V.

An easy example where $\xi_1^*(\omega_0, \omega(\beta)) \neq 0$ is given by $M = \mathbb{C} \setminus \{0\}$ with the flat Kähler metric, and by taking the unique component of β equal to $z \in M$. Then (3.21) gives $\xi_1^* = (\sqrt{-1}/2\pi)\,[dz/z] \neq 0$.

The higher order classes are much harder to express in the general case. Let us define

$$\kappa_k^{\bar h} = \eta_{\cdot \bar k}^{\bar l \cdot} D\beta_{\cdot \bar l}^{\bar h \cdot} = g^{\bar h u} g_{b \bar k} \eta_{a \cdot}^{\cdot b} D\beta_{u \cdot}^{\cdot a} \tag{3.22}$$

(which yields a tensor-valued 1-form on M), such that we may write

$$\xi_1^*(\omega_0, \omega(\beta)) = \frac{\sqrt{-1}}{2\pi}\,[\kappa_{\bar k}^{\bar k}]. \tag{3.23}$$

The curvature forms of the connection ∇ as given by (3.11) are

$$\begin{pmatrix} -\bar\Omega_k^l & 0 \\ 0 & -\bar\Omega_k^l \end{pmatrix}, \tag{3.24}$$

and then the curvature of $\tilde\nabla$ expressed by (3.20) is

$$\begin{pmatrix} D\kappa_{\bar k}^{\bar l} - \kappa_{\bar k}^{\bar h} \wedge \kappa_{\bar h}^{\bar l} - \bar\Omega_k^l & 0 \\ 0 & -\bar\Omega_k^l \end{pmatrix}, \tag{3.25}$$

where in (3.24) and (3.25) Ω_k^l are the curvature forms of D on M, and $D\kappa_{\bar k}^{\bar l}$ is the covariant exterior differential. Hence the forms Π_t of (2.5), (2.6) reduce to

$$\begin{pmatrix} -\bar\Omega_k^l + tD\kappa_{\bar k}^{\bar l} - t^2\kappa_{\bar k}^{\bar h} \wedge \kappa_{\bar h}^{\bar l} & 0 \\ 0 & -\bar\Omega_k^l \end{pmatrix}. \tag{3.26}$$

If these forms are introduced in (2.5), we obtain representative forms of the exotic classes $\xi_{2s+1}^*(\omega_0, \omega(\beta))$.

A nice formula is obtained only if $\bar\Omega = 0$ and $D\kappa_{\bar k}^{\bar l} = 0$, namely, $\xi_{2s+1}^*(\omega_0, \omega(\beta))$ are then represented by the differential forms (see [V2] for something similar in the real case)

$$\begin{aligned} \Xi_{2s+1}(\beta) = {} & \frac{\sqrt{-1}}{(2\pi)^{2s+1}(2s+1)!}\, \delta_{i_1 \cdots i_{2s+1}}^{j_1 \cdots j_{2s+1}} \\ & \cdot \kappa_{\bar j_1}^{\bar i_1} \wedge \kappa_{\bar j_2}^{\bar h_2} \wedge {}_{\bar h_2}^{\bar i_2} \wedge \cdots \wedge \kappa_{\bar j_{2s+1}}^{\bar h_{2s+1}} \wedge \kappa_{\bar h_{2s+1}}^{\bar i_{2s+1}}. \end{aligned} \tag{3.27}$$

It is important to notice that the classes $\xi_{2s+1}^*(\omega_0, \omega(\beta))$ are independent of the choice of the Kähler metric g on M since any other metric g' is homotopically related to g by the convex combination $(1-t)g + tg'$ $(0 \leqq t \leqq 1)$, and this yields a homotopy between the corresponding quaternionic structures.

Accordingly, we may summarize the results of this section as

THEOREM 3.1. *Let M^n be a complex manifold which has Kähler metrics and let β be a C^∞-automorphism of the vector bundle $T_{\mathrm{hol}}M$. Then (M, β) has an associated sequence of exotic characteristic classes $\xi^*_{2s+1}(\beta) \in H^{4s+1}(M, \mathbb{R})$ which do not depend on the choice of the Kähler metric on M, and vanish if β is homotopic to the identity via automorphisms of $T_{\mathrm{hol}}M$. $\xi^*_1(\beta)$ is represented by* (3.21), *and if M is Kähler flat and the associated form $\kappa^{\bar{l}}_{k}$ is parallel, the classes $\xi^*_{2s+1}(\beta)$ are represented by* (3.27).

Finally, let us notice that an endomorphism β as in Theorem 3.1 appears every time when M has a nondegenerate (1, 1)-form $\alpha(\alpha_{h\bar{k}})$. Indeed, then we may define $\beta^j_h = g^{\bar{k}j}\alpha_{h\bar{k}}$. Hence α has some related exotic classes on M. For instance, if M has a Kähler metric g with a nondegenerate and nonparallel Ricci form ρ, the exotic classes defined above are invariants of g itself, and they do not change if g is homotopically deformed within the class of Kähler metrics with a nondegenerate Ricci form. If the Ricci form ρ is positive definite, β is homotopic to the identity, and the exotic classes mentioned above are zero.

REFERENCES

[B] E. Bonan, *Sur les G-structures de type quaternionien*, Cah. Topol. Geom. Differ. **9** (1967), 389–461.

[D] R. Deheuvels, *Formes quadratiques et groupes classiques*, Presse Univ. France, Paris, 1981.

[GHV] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology II*, Academic Press, New York, 1973.

[KT] F. W. Kamber and Ph. Tondeur, *Foliated Bundles and Characteristic Classes*, Lecture Notes in Math. **493**, Springer-Verlag, New York, 1975.

[KN] S. Kobayashi and K. Nomizu, *Foundation of Differential Geometry II*, Interscience, New York, 1969.

[V1] I. Vaisman, *Cohomology and Differential Forms*, M. Dekker, New York, 1973.

[V2] I. Vaisman, *Symplectic Geometry and Secondary Characteristic Classes*, Progress in Math. **72**, Birkhäuser, Boston, 1987.